A Note on Countability

Richard L. Epstein

Recently I had a short conversation that helped bring into focus different views of the nature of mathematics. My student Esperanza Buitrago-Díaz and I went to the airport to pick up Henrique Antunes Almeida for his first visit to my research institute, the Advanced Reasoning Forum. On the way home I mentioned that we have sheep there.

> Henrique: How many sheep do you have?
> Me: I don't know. Every time I try to count them I fall asleep. . . . They're uncountable.
> Esperanza: That's ridiculous. That's not uncountable. I can count them.

I was taking "countable" to mean: can be counted by me. That's a completely subjective, personal standard. So my sheep are uncountable.

Perhaps, though, countable should mean: can be counted by almost anyone. That's an intersubjective standard. It would seem that by that standard my sheep are countable. But I'm not sure, since my friend's six-year old can't count them. We'd have to give some clearer description of what we mean by "almost anyone" for this to be a useful definition. And then we'd have to add that they (almost always) get the same number.

Such an intersubjective standard would make the number of stars visible in the night sky at my ranch January 6, 2014, uncountable. Yet we feel that those are countable. But by what standard? You might say that it's because there aren't an infinite number of them, to which I'd respond: How do you know? If you can't count them, then there might be an infinite number of them: there's no functional difference. Perhaps you could invoke some analysis in physics to show that there couldn't be an infinite number of them. Would that make it clear that they are countable?

It seems that a different standard is being invoked: can in theory be counted. That "in theory" is the refuge of someone who wants to keep the notion of countability close to what we can do, but without our poor human limitations. It must mean something like: a person who had unlimited time and attention and memory could count them. That's a considerable abstraction from our own capacities, yet still grounded in abstracting from those. This is the kind of definition Turing proposed and we now accept for the notion of computability.

Or we could say that our capacities are of no concern in defining what "countable" means—not what *we* mean but what it does mean, the reality of it. And that is simply: there is a one-to-one, onto function from either an initial segment of the natural numbers or from the entire set of natural numbers to that collection. This gives a definition that is clear. It assumes, however, some kind of abstract, impersonal notion of natural number and counting.

Sometimes that fully abstract definition is made a little closer to human abilities: there is a constructive one-to one, onto function from either an initial segment of the natural numbers or from the entire set of natural numbers. That however, requires a supplementation to the notion of computability to cover functions to or from collections other than numbers.

Only two definitions of "countable" are clear, entirely precise, and require no supplementation: the fully subjective and the fully abstract. The former is clear but does not lead to a shareable standard. The latter is clear but removes the notion from our lives, except to the extent that we can actually construct an enumerating function.

These different standards reflect different conceptions about how we do or should do mathematics, conceptions which have been adopted by both mathematicians and philosophers, though I shall not try to enumerate those here.