An Introduction to Formal Logic

Second Edition

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Preface to the Instructor

Logic is a tool we use to investigate how our language connects to the world so that we can reason better and, we hope, understand the world better. This book tells that story. It has a beginning, a middle, and an end.

We begin by setting out what formal logic is: the study of inferences for validity based on their form. Classical propositional logic is then presented as the simplest formal logic. In the development of that logic, the most important tools of formal logic are presented: a formal language, realizations, models, formal semantic consequence, and an axiom system. Examples of formalizing ordinary-language propositions and inferences show how to use classical propositional logic and also show some of its limitations.

The middle of the book relates form to the world. Predicate logic is motivated as a way to widen the scope of classical propositional logic to investigate more kinds of inferences whose validity depends on form. The large assumption that the world is made up of individual things is the basis for both the syntax and semantics of predicate logic. Besides form and the truth or falsity of atomic propositions, only the idea of assigning reference to terms as a kind of naming is needed. Many examples of formalizing show both the scope and limitations of classical predicate logic. Those depend on establishing criteria for what counts as a good formalization. The emphasis in those discussions is how the assumption that the world is made up of individual things is both useful and limiting.

The final chapter reflects on the success of the formal methods we've developed as means to reason to truths. Our formal logics circumscribe what we mean by "individual thing", namely, what can be reasoned about in predicate logic. Our formal logics give us a way to be precise about how we understand possibilities, though only relative to the assumptions we make about form and meaning and what there is in the world.

This story gives the basics, the fundamentals of formal logic. Along the way I point out how the work here can be extended and modified to apply to a wider scope of what we can formalize from ordinary-language reasoning. The story is not finished. Not here, not elsewhere.

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Some Points about the Organization and Content

- Appendices

  The appendices contain material that is either more technical than many students want, or too philosophical for many students, or is supplementary to the main line of the story.

- The form of atomic wffs

  Rather than take "Ralph is a dog" as a wff, we separate the roles of names and predicates. Thus, we write "(— is a dog) (Ralph)". We have a choice between "(— lives in —) (Arf, New Mexico)" or "(— lives in New Mexico) (Arf)" depending on whether we take "New Mexico" to be a name of a thing. This leads the student to see more clearly the roles of names and predicates and is crucially important in extending classical predicate logic to allow for formalizing reasoning that involves relative adjectives and adverbs in *The Internal Structure of Predicates and Names*.

- Superfluous quantification in the formal language of predicate logic

  In most logic texts, the definition of a formal language for predicate logic allows for superfluous quantifications. The rationale for including such formulas is that it simplifies the definition of the formal language, allowing a definition of bound and free variables to be made later. But the disadvantage is that a formula such as "$\forall x$ ((— is a dog) (Ralph))" that would correspond to the nonsensical "For everything, Ralph is a dog" is deemed acceptable. The semantics for superfluous quantifiers treat that formula as equivalent to "Ralph is a dog", which can be true. That is not consonant with our normally treating nonsense as false in our reasoning, as you can see in "Truth and Reasoning" in my *Reasoning and Formal Logic*. The advantages of not allowing superfluous quantification, beyond ridding our semi-formal languages of nonsense, are significant: we need no axiom schemes for superfluous quantification, and many proofs about the language are simplified by no longer having to treat cases of superfluous quantification separately.

- Proof theory

  Hilbert-style axiomatizations of classical propositional logic, classical predicate logic, and classical predicate logic with equality are presented in the text, and their completeness proofs appear in an appendix. Natural deduction and other methods of proof can be left until those skills are needed.

- Functions

  Only in some parts of mathematics and science are functions used, and then partial functions are essential. To extend classical predicate logic to allow for formalizing reasoning with partial functions we would need to analyze how to reason with non-referring names, descriptive names, and descriptive functions. I do that in *The Internal Structure of Predicates and Names*.

- The History of Logic

  It's a big subject. It's complicated. And it's not illuminating at this level. I present a lot in my books *Propositional Logics*, *Predicate Logic*, *Computability*, and *Classical Mathematical Logic*.

- English and Formal Logic

  Some might object that the development of formal logic here is too closely tied to motivations and examples from English. But in slightly modified form the examples and motivation here will apply to reasoning in many other languages. If they do not serve, then that would be evidence that the notion of thing is not as deeply embedded in the other language, as I explain in *Predicate Logic* and "Nouns and Verbs" in *Language and the World: Essays New and Old*.

3 Classical Propositional Logic: Form

A. Propositional Logic and the Basic Connectives

We've looked at some inferences that are valid due to their form, and we've seen some whose forms look a lot like the valid ones but are weak. As well, some proposition are true due to their form. For example, we don't need to know anything about Dick and what he likes in order to know that the following is true:

Dick liked the movie or Dick didn't like the movie.

Should we go on, looking at one form after another to try to decide whether it guarantees validity or truth? Do we have any method beyond our intuition?

We can do better. We can try to make precise what forms we're looking at. Then we can say how we'll understand the forms so that we can have a general method for evaluating propositions and inferences.

Formal logic *Formal logic* is (i) the analysis of inferences for validity in terms of the structure of the propositions appearing in an inference, and (ii) the analysis of propositions for truth in terms of their structure.

Formal logic will be a tool we can use in the analysis of inferences and propositions. By itself it can't determine whether any particular reasoning is good. Nor can it help us evaluate strong inferences.

To begin, we'll follow up on what we did in the last chapter, looking at simple forms based on how we can combine propositions.

Propositional logic *Propositional logic* is formal logic where we ignore the internal structure of propositions except as they are built from other propositions in specified ways.

Let's start with four phrases for building up propositions from other propositions we saw in the last chapter: the *connectives* "and", "or", "not", and "if . . . then . . .". These will give us a basis to see the issues in classifying forms and understanding meanings to serve us when we later extend what we pay attention to in the *syntax* (forms) and *semantics* (meaning) of propositions.

We use these English connectives in many ways, some of which may be of no concern to us in our reasoning. We'd like to agree on how we'll understand them and how they contribute to reasoning. But if we continue to use those English words, we're likely to forget how we've focussed on just this or that meaning. So let's use some symbols.

symbol	*what it will be an abstraction of*
$\wedge$	and
$\vee$	or
$\neg$	it's not the case that
$\rightarrow$	if . . . then . . .

So a sentence we might study is "Ralph is a dog $\wedge$ dogs bark".

When we talk about words or symbols, we should use quotation marks, saying "$\wedge$" and "$\rightarrow$" are symbols. But too many quotation marks make a page look like a series of hen tracks. So let's agree that a formal symbol can name itself when confusion seems unlikely. So I can say $\wedge$ is a formal connective.

Here's some terminology we use with these formal connectives:

- The sentence formed by joining two sentences with $\wedge$ is called a *conjunction*. Each of the original propositions is a *conjunct* that we *conjoin* with the other.
- The sentence formed by joining two sentences with $\vee$ is called a *disjunction*. Each of the original propositions is a *disjunct* that we *disjoin* with the other.
- The sentence formed by joining two sentences with $\rightarrow$ is called a *conditional*. The proposition on the left is called the *antecedent* and the one on the right is called the *consequent*.
- The sentence formed by putting $\neg$ in front of a proposition is called the *negation* of that proposition. It is a *negation*.

Example 1: *If Ralph is a dog, then Ralph barks or Ralph howls.*
Analysis It seems that we should replace this English sentence with:

(a) Ralph is a dog $\rightarrow$ Ralph barks $\vee$ Ralph howls

But this is ambiguous, though the example isn't because of the comma. Rather than using commas, let's use parentheses to mark off phrases.

Ralph is a dog $\rightarrow$ (Ralph barks $\vee$ Ralph howls)

Example 2: *If George is a duck then Ralph is a dog and Dusty is a horse.*
Analysis The original is ambiguous. Should we replace it with:

George is a duck $\rightarrow$ (Ralph is a dog $\wedge$ Dusty is a horse)
or
(George is a duck $\rightarrow$ Ralph is a dog) $\wedge$ Dusty is a horse

Only by asking the person who said the example or guessing from context can we make a choice. But it's clear we need to make a choice.

Exercises

1. Classify each of the following as: a conjunction (specify its disjuncts); a disjunction (specify its disjuncts); a conditional (specify its antecedent and consequent); a negation (specify what it is a negation of), or none of these.
   a. Ralph is a dog $\wedge$ dogs bark
   b. Ralph is a dog $\rightarrow$ dogs bark
   c. $\neg$ cats bark
   d. Cats bark $\vee$ dogs bark
   e. Ralph is a dog.
   f. Cats are mammals and dogs are mammals.
   g $\neg$ cats bark $\rightarrow$ $\neg$ cats are dogs
   h. Cats aren't nice.
   i. It is possible that Ralph is a dog.
   j. Either Ralph is a dog or Ralph isn't a dog.
2. a. Give a sentence that is a negation of a conditional whose antecedent is a conjunction.
   b. Give a sentence that is a conjunction of disjunctions, each of whose disjuncts is either a negation or has no formal symbols in it.
3. Give an example of a use of "not" that can't be understood as "It's not the case that".

B. The Formal Language of Propositional Logic

We want to talk in general about propositions. For the simplest one, the ones in which no formal connective appears, let's use the symbols $p_0, p_1, \ldots$. These are the *propositional variables*. These will be the first level of our formal language

We get more complex propositions by combining two of these using one of $\wedge, \vee, \rightarrow$ or preceding one by $\neg$.

Level 1 $p_0 \quad p_1 \quad p_2 \quad p_3 \quad \ldots$

Level 2 $p_0 \wedge p_1 \quad p_7 \wedge p_1 \quad p_{83} \wedge p_{12} \quad p_4 \vee p_1 \quad p_0 \rightarrow p_1 \quad \neg p_5 \quad \ldots$

At the next level, we get more complex forms by combining two of any of the preceding levels using one of $\wedge, \vee, \rightarrow$ or preceding one by $\neg$:

Level 3 $(p_0 \wedge p_1) \vee p_5 \quad \neg (p_7 \wedge p_1) \quad (p_4 \vee p_1) \wedge p_5$
$(p_{83} \wedge p_{12}) \rightarrow (p_0 \rightarrow p_1)$

At each level we use one more formal connective to build a propositional form.

How can we specify exactly what is in each level? How do we say, in general, that we can join any two forms from the second level with $\rightarrow$? Let's use some symbols, not part of the formal language we're establishing, to stand for any of the formulas, at any level: A, B, C, $A_0, A_1, A_2, \ldots$. And instead of talking of p_i and p_j and natural numbers, let's use p, q, and r, to stand for any of the propositional variables. These are *metavariables*. We can then define the forms that we will look at with an *inductive definition*: We state what is the lowest level, and then we say that if we already have forms of some levels, we can create a new form by combining those with one of the four connectives, taking us up one level. This will give us all the forms of the propositions we'll be considering, a completely formal "language".

The formal language of propositional logic $L(p_0, p_1, \ldots, \neg, \rightarrow, \wedge, \vee)$

Vocabulary *propositional variables* $p_0 \ p_1 \ \ldots$

connectives $\neg \ \rightarrow \ \wedge \ \vee$

Punctuation *parentheses*) (

Grammar (*well-formed formulas—wffs*)

Each of (p_0), (p_1), (p_2), (p_3), . . . is a wff of *length* 1.

If A is a wff of length n, then $(\neg A)$ is a wff of *length* $n + 1$.

If A and B are wffs and the maximum of their lengths is n, then $(A \rightarrow B)$, $(A \wedge B)$, and $(A \vee B)$ are wffs of *length* $n + 1$.

A concatenation of symbols of the vocabulary is a *wff* iff it is a wff of length n for some $n \geq 1$.

Wffs of length 1 are *atomic*; all others are called *compound*.

It may seem obvious that parentheses ensure that if a concatenation of symbols is a wff, then there's only one way to read it. But that needs a proof. It's a proof by *induction*; if you're not familiar with that method, you read Appendix 1.

Theorem 1 The unique readability of wffs

There is one and only one way to parse each wff of $L(\neg, \rightarrow, \wedge, \vee, p_0, p_1, \ldots)$.

Proof To each primitive symbol α of the formal language, assign an integer $\lambda(\alpha)$ according to the following chart:

$\neg$	$\rightarrow$	$\wedge$	$\vee$	p_i	(	)
0	0	0	0	0	– 1	1

To the concatenation of symbols $\alpha_1 \alpha_2 \cdots \alpha_n$, assign the number:

$$\lambda(\alpha_1) + \lambda(\alpha_2) + \cdots + \lambda(\alpha_n)$$

We'll first show that for any wff A, $\lambda(A) = 0$, using induction on the number of occurrences of $\neg, \rightarrow, \wedge, \vee$ in A.

If there are no occurrences, then A is atomic, that is, for some $i \geq 0$, A is (p_i). Then $\lambda(`(`) = -1$, $\lambda(p_i) = 0$, and $\lambda(`)') = 1$. Adding, we have $\lambda(A) = 0$.

Suppose the lemma is true for every wff that has fewer occurrences of these symbols than A does. Then there are 4 cases, which we can't yet assume are distinct: A arises as $(\neg B)$, $(B \rightarrow C)$, $(B \wedge C)$, or $(B \vee C)$. By induction $\lambda(B) = \lambda(C) = 0$, so in each case by adding, we have $\lambda(A) = 0$.

Now I'll show that if α is an initial segment of a wff, reading from the left, other than the entire wff itself, then $\lambda(\alpha) < 0$; and if α is a final segment reading from the left other than the entire wff itself, then $\lambda(\alpha) > 0$. So no proper initial or final segment of a wff is a wff. To establish this I will again use induction on the number of occurrences of connectives in the wff. I'll let you establish the base case for atomic wffs, where there are no (that is, zero) connectives.

Suppose now that the lemma is true for any wff that contains $\leq n$ occurrences of the connectives. If A contains $n + 1$ occurrences, then it must have (at least) one of the forms given in the definition of wffs. If A has the form (B$\wedge$ C), then an initial segment of A must have one of the following forms:

i. (

ii. (β where β is an initial segment of B

iii. (B

iv. (B $\wedge$

v. (B $\wedge$ γ where γ is an initial segment of C

For (ii), $\lambda($'('$) = -1$, and by induction $\lambda(\beta) < 0$, so $\lambda($'(β'$) < 0$. I'll leave (i), (iii), (iv), and (v) to you. The other cases (for $\neg$, $\rightarrow$, and $\vee$) follow similarly, and I'll leave those and the proof for final segments to you.

Now to establish the theorem, we proceed through a number of cases by way of contradiction. Suppose we have a wff that could be read as both (A$\wedge$ B) and (C$\rightarrow$ D). Then A$\wedge$ B) must be the same as C$\rightarrow$ D). In that case either A is an initial part of C or C is an initial part of A. But then $\lambda(A) < 0$ or $\lambda(C) < 0$, which is a contradiction, as we proved above that $\lambda(A) = \lambda(C) = 0$. Hence, A is C. But then we have that $\wedge$ B) is the same as $\rightarrow$ D), which is a contradiction.

Suppose ($\neg$A) could be parsed as (C$\rightarrow$ D). Then ($\neg$A and (C$\rightarrow$ D must be the same. So D would be a final segment of A other than A itself, but then $\lambda(D) > 0$, which is a contradiction. The other cases are similar, and I'll leave them to you. ■

Example 3: (p_1) has length 1

Example 4 : $(\neg (p_1))$ has length 2

Example 5: $((\neg (p_1)) \wedge (p_2))$ has length 3

Example 6: $(((\neg (p_1)) \wedge (p_2)) \rightarrow (p_3))$ has length 4.

$$\underbrace{(\underbrace{(\underbrace{(\neg\,\underbrace{(p_1)}_{1})}_{2} \wedge \underbrace{(p_2)}_{1})}_{3} \rightarrow \underbrace{(p_3)}_{1})}_{4}$$

It's hard to read a long wff because of all the parentheses. Those are needed to ensure that there is just one way to read each wff. But informally we can delete some of them if we adopt some conventions:

- We can drop the outermost parentheses.
- We can drop the parentheses around the propositional variables.

- We'll say that $\neg$ binds more strongly than $\wedge$, $\vee$, or $\rightarrow$.
- We'll say that $\wedge$ and $\vee$ bind equally strongly, but both more strongly than $\rightarrow$.

Example 7: $\neg\, p_1 \wedge\ p_2$ abbreviates $((\neg (p_1) \wedge\ (p_2))$

Example 8 : $(p_1 \wedge\ p_2) \vee\ p_3$ abbreviates $(((p_1) \wedge (p_2)) \vee\ (p_3))$

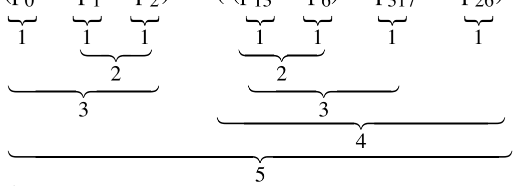
Example 9 : $\neg\, p_1 \wedge\ p_2 \rightarrow\ p_3$ abbreviates $(((\neg (p_1)) \wedge (p_2)) \rightarrow\ (p_3))$

These are informal abbreviations to make it easier to write and read formulas. An abbreviated wff is not a formal wff.

Example 10: $(((p_0) \rightarrow ((p_1) \wedge (p_2))) \rightarrow\ ((((p_{13}) \wedge (p_6)) \vee (p_{317})) \rightarrow (p_{26})))$

Analysis Abbreviating this wff , we can see it has length 5.

$(p_0 \rightarrow\ p_1 \wedge\ p_2) \rightarrow\ (\,(p_{13} \wedge\ p_6) \vee\ p_{317} \rightarrow\ p_{26})$



Exercises

1. Why do we introduce a formal language?
2. Identify which of the following are formal (unabbreviated) wffs:
   a. $(p_1) \vee \neg (p_2)$
   b. $((p_1) \rightarrow (p_2))$
   c. $((p_1 \vee p_2) \rightarrow p_2)$
   d. $(\neg (p_1)\ (p_2) \wedge (p_1))$
   e. $(\neg (\neg (p_1))\ \vee \neg (p_1))$
   f. $((\neg (\neg (p_1))) \vee \neg (p_1))$
   g. $((\neg (\neg (p_1))) \vee (\neg (p_1)))$
3. Abbreviate the following wffs according to our conventions and then give the length of each.
   a. $(((\neg p_0) \wedge p_{13}) \rightarrow p_2)$
   b. $((((p_1) \rightarrow (p_2)) \wedge (\neg (p_2))) \rightarrow\ (\neg (p_1)))$
   c. $((((p_4)\ \wedge\ (p_2)) \vee (\neg (p_6))) \rightarrow\ ((p_7) \rightarrow (p_8)))$
4. Give an example (abbreviated is O.K.) of a wff that is:
   a. A conjunction, the conjuncts of which are disjunctions of either propositions or negated propositions.
   b. A conditional whose antecedent is a disjunction of negations and whose consequent is a conditional whose consequent is a conditional.

Aside: *Defining a formal language without induction*

Some logicians define the formal language of propositional logic as:

> $L(\neg, \rightarrow, \wedge, \vee, p_0, p_1, \ldots)$ is the smallest collection containing (p_i) for each $i = 0, 1, 2, \ldots$ and closed under the formation of wffs, that is, if A, B are in the collection, so are $(\neg A)$, $(A \rightarrow B)$, $(A \wedge B)$, and $(A \vee B)$.

One collection is said to be smaller than another if it is contained in or equal to the other.

This definition seems to require us to accept that there are completed infinite collections. Yet when these logicians want to prove anything about the formal language, they still have to give a definition of the length of a wff, and that amounts to giving an inductive definition of the language in addition to this one.

C. Realizations and Semi-Formal Languages

The formal language gives us the forms of the propositions we will study. A wff such as $p_0 \wedge \neg p_1$ is neither true nor false; it is the skeleton of a proposition. Only when we fix on a particular interpretation of the formal connectives and then assign propositions to the variables, such as "p_0" stands for "Ralph is a dog" and "p_1" stands for "Four cats are sitting in a tree", do we have a semi-formal proposition, "Ralph is a dog $\wedge$ $\neg$ (four cats are sitting in a tree)" that can be viewed as having a truth-value. We may read this as "Ralph is a dog and it's not the case that four cats are sitting in a tree" so long as we remember that we've agreed that all "and" and "it's not the case that" mean will be captured by the interpretations we'll give for $\wedge$ and $\neg$.

Realizations and semi-formal languages A *realization* is an assignment of propositions to some or all of the propositional variables. The *realization of a formal wff* is the formula we get when we replace the propositional variables appearing in the formal wff with the propositions assigned to them; it is a *semi-formal wff*. The *semi-formal language* for the realization is the collection of realizations of the formal wffs.

Example 11: Here is a realization:

p_0	"Ralph is a dog"
p_1	"Four cats are sitting in a tree"
p_2	"Four is a lucky number"
p_3	"Dogs bark"
p_4	"Juney is barking loudly"
p_5	"Juney is barking"
p_6	"Dogs bark"
p_7	"Ralph is barking"
p_8	"Cats are nasty"
p_9	"Ralph barks"
p_{47}	"Howie is a cat"
p_{312}	"Bill is afraid of dogs"
p_{317}	"Bill is walking quickly"
p_{4318}	"If Ralph is barking, then he will catch a cat"
p_{4319}	"Ralph is barking"

Analysis We don't need to realize all the propositional variables. And we can assign the same proposition to more than one variable. In this example, the realization of p_7 is the same as the realization of p_{4319}. And the realization of: $((p_0) \rightarrow ((p_1) \wedge (p_2)))$ is:

((Ralph is a dog) $\rightarrow$ ((Four cats are sitting in a tree) $\wedge$ (Dogs bark)))

The propositions we assign to the propositional variables are meant to be the simplest ones we can begin with. We say they're *atomic* because we are not concerned with their internal structure, since they involve no logical symbols. So the assignment of "If Ralph is barking, then he will catch a cat" to p_{4318} is a bad choice, for it will not allow us to make an analysis based on the form of that proposition in relation to what we assign to p_7, "Ralph is barking".

Semi-formal wffs inherit the terminology of the formal wffs they realize: atomic, compound, conjunction, etc. We can use the same metavariables A, B, C, $A_0, A_1, A_2, \ldots$ for formal wffs, semi-formal wffs, or propositions in English, trusting to context to make clear which are meant.

Exercises

1. For Example 11, give the realization of each of the following (abbreviated) wffs.
   a. $((p_8 \wedge p_{4318}) \wedge p_7) \rightarrow p_1$
   b. $(p_0 \wedge p_1) \rightarrow p_2$
   c. $\neg(p_4 \wedge \neg p_5)$
   d. $p_3 \rightarrow \neg\neg p_6$
   e. $\neg(p_{312} \wedge p_7) \wedge \neg p_{317}$
   f. $p_{312} \wedge p_7 \rightarrow p_{317}$
2. The following wffs are (an abbreviations of) the realization of what formal wffs in the realization of Example 11?
   a. Ralph is a dog $\wedge$ $\neg$ (four cats are sitting in a tree)
   b. Bill is afraid of dogs $\wedge$ Ralph barks $\rightarrow$ Bill is walking quickly
   c. Juney is barking loudly $\rightarrow$ Juney is barking
   d. Ralph is a dog $\wedge$ dogs bark $\rightarrow$ Ralph barks
   e. Four cats are sitting in a tree $\wedge$ four is a lucky number $\rightarrow$ $\neg$ (Dogs bark $\rightarrow$ Howie is a cat)

Key Words		
	formal logic	compound wff
	propositional logic	unique readability of wffs
	$\wedge$, $\vee$, $\neg$, $\rightarrow$	realization
	formal language	semi-formal language
	atomic wff	